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ON THE MINIMUM SURFACE OF REVOLUTION IN THE CASE OF ONE VARIABLE END POINT

By Mary Emily Sinclair

Introduction. In the problem of minimizing the integral

$$J = \int_{x_0}^{x_1} y\sqrt{1 + y'^2}\,dx,$$

that is, of finding a minimum surface of revolution when the end point $P_1(x_1, y_1)$ of the generating curve is fixed and the end point $P_0(x_0, y_0)$ is permitted to vary along a curve

$$D: \quad y = g(x),$$

the solution of Euler's differential equation gives the set of extremals *

$$(1) \qquad E. \qquad y = a\,\mathrm{Ch}\,\frac{x - b}{a}.$$

We determine b as a function of a,

$$b = x_1 - a\,\mathrm{Ch}^{-1}\,\frac{y'}{a},$$

by the condition that the curve (1) passes through P_1, thus obtaining the one-parameter set of extremals through P_1,

$$(2) \qquad y = a\,\mathrm{Ch}\,\frac{x - b(a)}{a}.$$

The condition of transversality † gives, for our problem, at the point P_0,

$$(3) \qquad 1 + y'g' = 0,$$

that is, the extremal must be perpendicular to the curve D.

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* We use Laisant's notation for hyperbolic functions.

† Kneser, *Lehrbuch der Variationsrechnung*, §10. Bolza, *Lectures on the Calculus of Variations*, §23, (33).

Assuming that (3) is fulfilled for at least one extremal E_0 of the set E, we denote by P_0 the point common to D and E_0. It is then further necessary that the focal point of the curve D shall not lie between P_0 and P_1. Any of

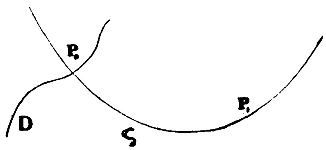

FIG. 1.

the general formulae * give for the determination of the focal point, $\bar{P}(\bar{x}, \bar{y})$, the following equation : †

$$(4) \qquad \frac{\phi(\bar{u}) - \phi(u_0)}{\phi(\bar{u}) + \operatorname{Sh} u_0 \operatorname{Ch} u_0 + u_0} = - \frac{\rho}{\epsilon a \operatorname{Ch}^2 u_0 \operatorname{Sh} u_0},$$

where
$$\bar{u} = \frac{\bar{x} - b}{a}, \qquad u_0 = \frac{x_0 - b}{a},$$

$$\phi(u) = \operatorname{Coth} u - u,$$

$$\epsilon = \frac{\operatorname{Sh} u_0}{|\operatorname{Sh} u_0|},$$

and ρ is the radius of curvature of D at P_1 with the usual agreement as to sign.‡ It is evident that $\epsilon = \pm 1$ as $u_0 \gtrless 0$.

In this paper we propose (1) to give a detailed *discussion of the transcendental equation* (4) for the determination of the focal point, (2) to obtain a *simple geometric construction for the focal point* which shall include the Lindelöf§ construction as a special case when $\rho = 0$, and (3) to give a *physical interpretation of the focal point* as defining for the case $\rho = \infty$ the limit of stability of a liquid film.‖

* Kneser, loc. cit., §23. Bolza, loc. cit., §23. Bliss, *Transactions of the Amer. Math. Soc.*, vol. 3 (1902), p. 132.

† Given in slightly different form by Kneser, loc. cit , p. 85, equation (65).

‡ Compare, for instance, Scheffers, *Anwendungen der Differential- und Integralrechnung auf Geometrie*, vol. 1, p. 37.

§ See, for instance, Bolza, loc. cit., p. 64.

‖ See Plateau, *Statiques des liquides*, in particular §90, 1 and §§111, 225–227.

1. Discussion of the transcendental equation (4). In order to discuss equation (4), we shall first study the function

(5) $$\phi(u) = \operatorname{Coth} u - u.$$

Since $\phi'(u) = -\operatorname{Coth}^2 u < 0$, $\phi(u)$ is a decreasing function of u. Also

$$\phi''(u) = \frac{2\operatorname{Ch} u}{\operatorname{Sh}^3 u} \gtrless 0$$

according as $u \gtrless 0$. Further $\phi(u)$ becomes infinite if and only if $u = 0$ or ∞. Hence, $\phi(u)$ is a continuous, decreasing function which takes every value k once and but once for $u > 0$ and once and but once for $u < 0$. The curve is as shown in figure 2.

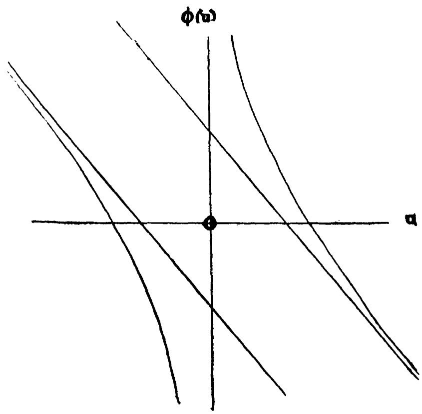

FIG. 2.

We now consider ρ as a function of $\bar{u}$, writing (4) as follows:

(6) $$\rho = -\epsilon a \operatorname{Ch}^2 u_0 \operatorname{Sh} u_0 \frac{\phi(\bar{u}) - \phi(u_0)}{\phi(\bar{u}) - \phi(v_0)},$$

where v_0 and v_0' are the positive and negative quantities respectively, defined by the equations

(7) $$\phi(v_0) = \phi(v_0') = -(u_0 + \operatorname{Ch} u_0 \operatorname{Sh} u_0).$$

Then $$\frac{d\rho}{d\bar{u}} = -\epsilon a \operatorname{Ch}^5 u_0 \phi'(\bar{u}) \frac{1}{[\phi(\bar{u}) - \phi(v_0)]^2}.$$

We assume throughout $a > 0$, and consider two cases, $u_0 < 0$ and $u_0 > 0$.

Case 1. $u_0 < 0$. Then $\epsilon = -1$, and therefore $d\rho/d\bar{u} < 0$, whence ρ is everywhere a decreasing function of $\bar{u}$. Further, ρ is a linear function of $\phi(\bar{u})$, and therefore takes every value twice as $\bar{u}$ varies from $-\infty$ to $+\infty$. We shall denote by u_0' the value of $\bar{u}$ at P_0', the point conjugate to P_0, u_0' being determined by the equation,

$$\phi(u_0') - \phi(u_0) = 0.*$$

From (6) the values taken by $\bar{u}$ when $\rho = \infty$ are v_0 and v_0', where $v_0 < u_0$,

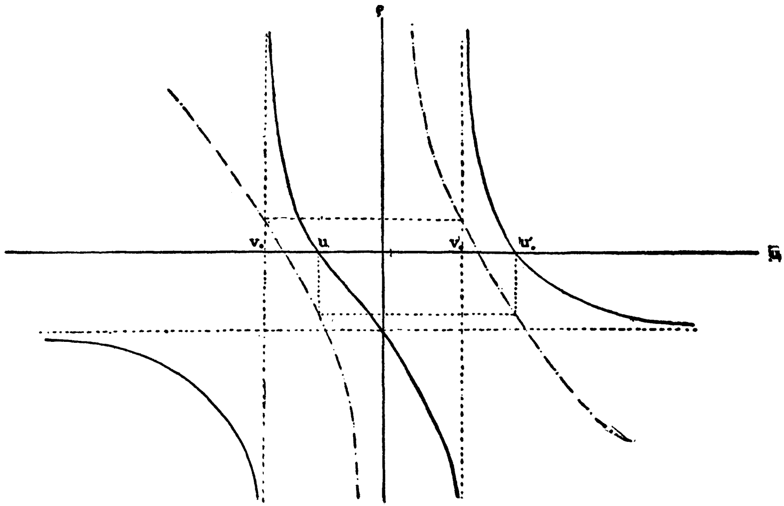

FIG. 3.

$v_0' < u_0'$, since from (5) and (7) $\phi(v_0) > \phi(u_0)$ and $\phi(v_0') > \phi(u_0')$. We obtain, then, the following table and the curve in figure 3:

$\bar{u}$	$-\infty$	v_0	u_0	0	v_0'	u_0'	$+\infty$
$\phi(\bar{u})$	$+\infty$	$\phi(v_0)$	$\phi(u_0)$	$\mp\infty$	$\phi(v_0)$	$\phi(u_0)$	$-\infty$
ρ	M	$\mp\infty$	0	M	$\mp\infty$	0	M,

where $M = a\,\mathrm{Ch}^2 u_0\,\mathrm{Sh}\,u_0$.

As $\bar{u}$ increases from u_0 to u_0', ρ decreases continuously from 0 to $-\infty$ and again from $+\infty$ to 0. Conversely, for every value of ρ between $-\infty$ and $+\infty$ there exists one and only one corresponding value of $\bar{u}$ between u_0 and u_0'. Remembering that $x = au + b$, we conclude that as ρ decreases from 0 to

* See, for instance, Bolza, loc. cit., p. 64, (32).

$-\infty$ and again from $+\infty$ to 0, $\bar{x}$ increases continuously from x_0 to x_0' and therefore $\bar{P}_0$ describes the catenary in positive sense from P_0 to P_0'. Hence, every value of ρ determines one and only one point of E_0 focal to the curve D and lying on the interval $P_0\,P_0'$.*

Case 2. $u_0 > 0$. Here $\mathrm{Sh}\,u_0 > 0$ and $\epsilon = +1$, and therefore $d\rho/d\bar{u} > 0$ and ρ is everywhere an increasing function. The root u_0' of the equation $\phi(u) - \phi(u_0) = 0$ is negative, and therefore less than u_0. Hence in this case there exists no point conjugate† to P_0 with an abscissa greater than that of P_0. The values taken by $\bar{u}$ when $\rho = \infty$ are x_0 and v_0' where $v_0 > u_0$ and $v_0' > u_0'$, since from (5) and (7) again $\phi(v_0) < \phi(u_0)$ and $\phi(v_0') < \phi(u_0')$. We obtain then the following table:

$\bar{u}$	$-\infty$	u_0'	v_0'	0	u_0	v_0	$+\infty$
$\phi(\bar{u})$	$+\infty$	$\phi(u_0)$	$\phi(v_0)$	$\mp\infty$	$\phi(u_0)$	$\phi(v_0)$	$-\infty$
ρ	$-M$	0	$\pm\infty$	$-M$	0	$\pm\infty$	$-M$,

where $M = a\,\mathrm{Ch}^2 u_0\,\mathrm{Sh}\,u_0$.

We study the interval $u_0 \cdots \infty$. As $\bar{u}$ increases from u_0 to ∞, ρ increases from 0 to $+\infty$ and again from $-\infty$ to $-a\,\mathrm{Ch}^2 u_0\,\mathrm{Sh}\,u_0$. Conversely, to

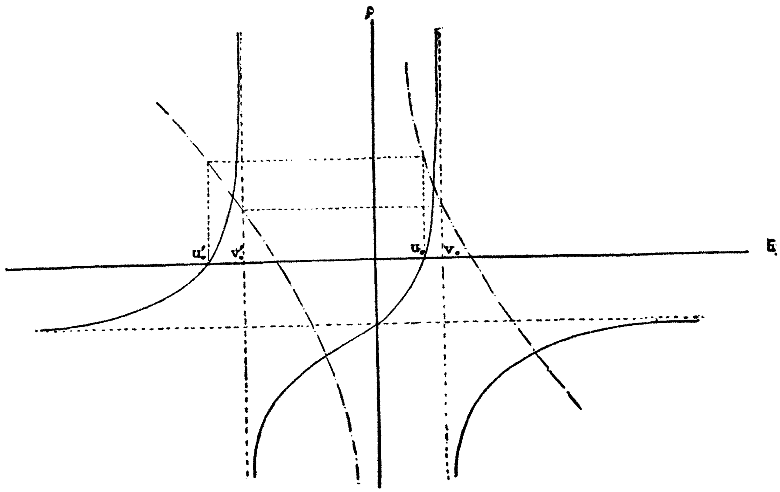

FIG. 4.

* In accordance with a general theorem due to Bliss, *Trans. Amer. Math. Soc.*, vol. 3 (1902), p. 139.

† See Kneser, loc. cit., p. 89. Bolza, loc. cit., §14.

every value of ρ between $-a\,\mathrm{Ch}^2 u_0\,\mathrm{Sh}\,u_0$ and 0 there exists no corresponding $\bar{u}$ on the interval, but for all other values of $\bar{u}$ between $-\infty$ and $+\infty$, there exists one and only one corresponding value of $\bar{u}$ on the interval (see figure 4). As ρ increases from 0 to $+\infty$ and from $-\infty$ to $-a\,\mathrm{Ch}^2 u_0\,\mathrm{Sh}\,u_0$, $\bar{x}$ increases continuously from x_0 to ∞, and therefore $\bar{P}_0$ describes the catenary in positive sense from P_0 to ∞. When ρ lies between $-a\,\mathrm{Ch}^2 u_0\,\mathrm{Sh}\,u_0$ and v, there exists no focal point $\bar{P}_0$ with an abscissa greater than that of P_0.

2. A Geometrical construction for the focal point. We proceed to obtain a geometric construction for the focal point P_0, treating separately the two cases discussed above.

Case 1. $u_0 < 0$. From (6) we have:

$$\frac{a[\phi(\bar{u}) - \phi(v_0)]}{a[\phi(u_0) - \phi(v_0)]} = -\frac{a\,\mathrm{Ch}^2 u_0\,\mathrm{Sh}\,u_0}{\rho - a\,\mathrm{Ch}^2 u_0\,\mathrm{Sh}\,u_0}. \tag{8}$$

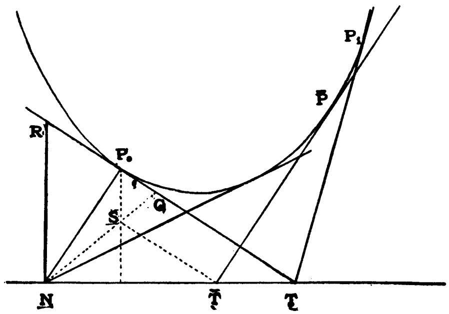

FIG. 5.

Let the tangent and normal at P_0 cut the x axis at T and N respectively, and let R be the point on the tangent whose abscissa is that of N.

Then $$NT = -a\,[\phi(u_0) - \phi(v_0)]$$

and $$|P_0R| = -a\,\mathrm{Ch}^2 u_0\,\mathrm{Sh}\,u_0.$$

Further if $\bar{T}$ is the point of intersection of the tangent at the focal point $\bar{P}_0$ with the x-axis, then

$$N\bar{T} = -a\,[\phi(\bar{u}) - \phi(v_0)].$$

Equation (8) then becomes

$$\frac{N\bar{T}}{NT} = \frac{|P_0R|}{\rho + |P_0R|}, \tag{9}$$

and $\bar{T}$ may now be located for any assigned value of ρ by the following construction :—

On P_0T lay off $P_0Q = \rho$. Draw NQ and let the ordinate at $\bar{P}$ intersect NQ in S. Draw $S\bar{T}$ parallel to RT. Of the two tangents which can be drawn from $\bar{T}$ to the catenary, one will meet the curve at a point $\bar{P}_0$ between P_0 and P_0', and $\bar{P}_0$ is the focal point.

Three special cases are of interest.

(a) $\rho = 0$. Here P_0 coincides with S, $\bar{T}$ with T, and $\bar{P}_0$ with P_0'. The construction reduces to the Lindelöf * construction for the conjugate of P_0, the equation being the well-known one,

$$\phi(\bar{u}) - \phi(u_0) = 0.$$

(b) $\rho = \infty$. Here $\bar{T}$ coincides with N, the equation being

$$\phi(\bar{u}) - \phi(v_0) = 0,$$

and we obtain the following simple rule :

When $\rho = \infty$, $\bar{P}_0$ is the point of tangency of that tangent from N which meets the curve between P_0 and P_0'.

(c) $\rho = a \operatorname{Ch}^2 u_0 \operatorname{Sh} u_0$. Here Q coincides with R, and $\bar{T}$ is at an infinite distance on the x-axis. The focal point is therefore the vertex of the catenary.

Now let ρ decrease continuously from 0 to $-\infty$ and again from $+\infty$ to 0. $\bar{T}$ then describes the x-axis in positive sense, starting from T when $\rho = 0$ and passing through the infinite point of the x-axis when $\rho = a \operatorname{Ch}^2 u_0 \operatorname{Sh} u_0$, and through N when $\rho = \mp \infty$, returning to T for the value $\rho = 0$. At the same time $\bar{P}_0$ describes $P_0 P_0'$ in positive sense.†

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* Lindelöf, *Mathematische Annalen*, vol. 2 (1870), p. 160. Bolza, loc. cit., p. 64.

† I have generalized these results for the case

$$J = \int_{x_0}^{x_1} y^k \sqrt{1 + y'^2}\, dx.$$

Euler's equation gives here

$$y' = \sqrt{\left(\frac{y}{a}\right)^{2k} - 1}.$$

Denoting the solution of this equation by $y = a\,\phi(u)$ where $u = (x - b)/a$, we obtain

Case 2. $u_0 > 0$. In this case, (6) becomes

$$(8') \qquad \frac{a[\phi(\bar{u}) - \phi(v_0)]}{a[\phi(u_0) - \phi(v_0)]} = \frac{a\,\mathrm{Ch}^2 u_0\,\mathrm{Sh}\,u_0}{\rho + a\,\mathrm{Ch}^2 u_0\,\mathrm{Sh}\,u_0}.$$

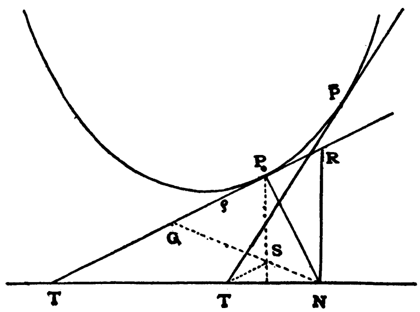

FIG. 6.

Let the tangent and normal at P_0 cut the x-axis at T and N respectively, and let R be the point on the tangent whose abscissa is that of N.

Then
$$NT = -a[\phi(u_0) - \phi(v_0)],$$

and
$$|P_0 R| = a\,\mathrm{Ch}^2 u_0\,\mathrm{Sh}\,u_0.$$

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for the determination of the focal point the equation

$$\frac{-a\,\phi_0\phi_0'(1+\phi_0'^2)^{1/2}}{k\rho} = \frac{\dfrac{\bar{\phi}}{\bar{\phi}'} - \bar{u} + u_0 + \phi_0\phi_0'}{\dfrac{\bar{\phi}}{\bar{\phi}'} - \bar{u} - \dfrac{\phi_0}{\phi_0'} + u_0},$$

where ϕ_0 and u_0 are used to refer to the initial point and $\bar{\phi}$ and $\bar{u}$ for the focal point. Interpreted geometrically, this equation is:

$$-\frac{|P_0R|}{k\rho} = \frac{N\bar{T}}{N\bar{T} - NT},$$

whence
$$\frac{N\bar{T}}{NT} = \frac{|P_0R|}{\rho k + |P_0R|}$$

If therefore, $k\rho$ is used in place of ρ in the construction given in the text, $\bar{T}$ is determined in this more general case, and the point of contact of the tangent from $\bar{T}$ to the extremal is again the focal point.

And, if $\bar{T}$ is again the point of intersection of the tangent at the focal point with the x-axis, then $N\bar{T} = -a[\phi(\bar{u}) - \phi(v_0)]$. Equation (8′) becomes

$$\frac{N\bar{T}}{NT} = \frac{|P_0R|}{\rho + |P_0R|}, \tag{8}$$

and $\bar{T}$ may now be located for any assigned value of ρ by the same construction as in case 1. *When $u_0 > 0$ the construction of the point T is similar to that given above, but of the two tangents which can be drawn from $\bar{T}$ to the catenary, that one must be chosen which meets the curve at a point $\bar{P}_0$ whose abscissa is greater than that of P_0. $\bar{P}_0$ is then the focal point.* Such a point always exists for all values of ρ except values $-a\,\mathrm{Ch}^2 u_0 \,\mathrm{Sh}\, u_0 < \rho < 0$, as we showed in §1.

3. Physical illustration for the case $\rho = \infty$. The theory of minimum surfaces may be beautifully illustrated by experiments with liquid films. An extended discussion of such experiments is to be found in Plateau's treatise. The following simple experiment* illustrates the theory of the focal point. A glass funnel A is held in upright position and a smaller one B is inverted within it so that its rim is in contact with the inner surface of A. Soap solution is then applied at this rim, and the funnel B is withdrawn vertically, care being taken to maintain an axis of symmetry. A cathetometer is used for the necessary measurements. A catenoid film is formed, extending from the lateral surface of the funnel A to the rim of B. Its lower opening creeps up the inner surface of the funnel A, equilibrium being found when the angle between the film and the funnel A is 90°. The film is stationary and

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* See Plateau's experiments, loc. cit. Two wire rings are placed in contact, moistened with soap solution, and drawn apart. A catenoid film then extends from one to the other, and possesses perfect stability up to a certain limit, the length of axis being then about two thirds the diameter of the rings. Plateau's experimental results are compared below with those of Lindelöf which are given by the theory of the conjugate point [see *Math. Ann.*, vol. 2 (1870) p. 160].

	diameter of ring.	length of axis.	diameter of neck.
Theoretical value	71.49	47.38	39.47
Experimental value	71.49	46.85	39.60

A second experiment of Plateau's determines the limit of stability of a catenoid film formed by withdrawing vertically one ring from the surface of a soap solution. In this case the limit of stability is given by the theory of the focal point, when the curve D is a straight line parallel to the y-axis.

perfectly stable when any fixed height is maintained, so long as this height is within a certain limit. At the limit of stability, which occurs when the point P in figure 7 is the focal point of the generator D of the funnel A, the film gradually separates at the neck into two convex films which recede respectively upon the two funnels. Beyond this limit, the catenoid is not the surface of least area.

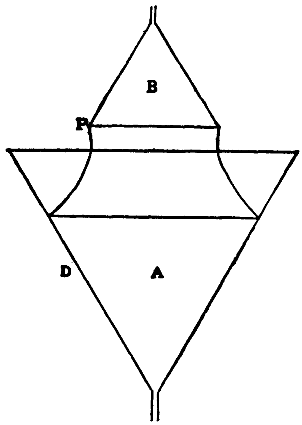

FIG. 7.

We shall give the mathematical solution of the problem and then compare it with experimental results. Since all catenaries are similar figures, we obtain the values when $a = 1$, and then adapt our values to the constants of the apparatus. As x-axis we use the axis of symmetry of the apparatus and as y-axis a perpendicular through the vertex of A. Let α be the semi vertical angle of A. Let P_0 be a point common to the catenary E and the line D, whose equations are as follows:

$$(10) \qquad D: \quad \begin{cases} x = r \cos \alpha, \\ y = r \sin \alpha; \end{cases}$$

$$(11) \qquad E: \quad \begin{cases} x = u + b, \\ y = \operatorname{Ch} u. \end{cases}$$

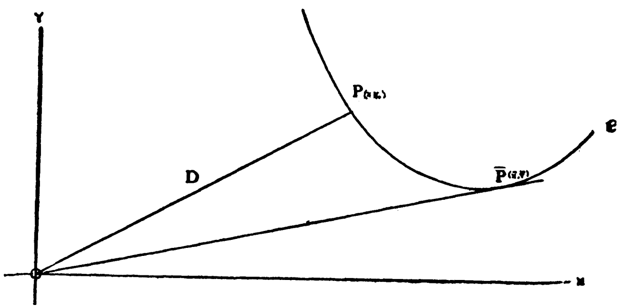

FIG. 8.

The point $\bar{P}_0$ is then determined by a tangent through the origin. We seek to determine x_0, y_0, b, $\bar{x}_0$, $\bar{y}_0$.

For E,

$$y' = \operatorname{Sh} u = \epsilon \sqrt{y^2 - 1},$$

where $\epsilon = \dfrac{|\operatorname{Sh} u|}{\operatorname{Sh} u}$ and the sign $\sqrt{\ }$ is used always for the positive square root. Since, by the condition of transversality, E and C are perpendicular at P_0, we have

$$\cos \alpha - \sin \alpha \sqrt{y_0^2 - 1} = 0. \quad \text{Hence}$$

$$(12) \qquad y_0 = \csc \alpha, \qquad r_0 = \csc^2 \alpha, \qquad \bar{x}_0 = \cos \alpha \csc^2 \alpha.$$

Therefore
$$\csc a = \mathrm{Ch}\left(\frac{\cos a}{\sin^2 a} - b\right),$$
or

(13)
$$b = \frac{\cos a}{\sin^2 a} - \mathrm{Ch}^{-1}(\csc a)\cdot$$

Since $\bar{P}_0$ is determined by the tangent from the origin, we have

(14)
$$\mathrm{Coth}\,\bar{u}_0 = \bar{x}_0 = \bar{u}_0 + b.$$

If we can solve this transcendental equation we may determine $\bar{y}_0$ from $\bar{y}_0 = \mathrm{Ch}\,\bar{u}_0$. Equation (14) may be written, using the notation of §1, $\phi(\bar{u}) = b$, which we have shown admits of a unique positive solution $\bar{u}_0$.

Let us now obtain the values of the constants when $a = 30°$, the actual angle of the experiment.

From (12) we obtain $y_0 = 2$, $x_0 = 2\sqrt{3} = 3.464$, $u_0 = \mathrm{Ch}^{-1}2 = -1.317$, $b = 2\sqrt{3} - \mathrm{Ch}^{-1}2 = 4.781$, since u_0 is negative.

We now obtain $\bar{u}_0$ from the equation $\phi(\bar{u}_0) = 4.781$ by approximation.

u	e^{2u}	Coth u	$\phi(u)$
.200	1.492	5.070	4.870
.203	1.5004	4.9970	4.791
.204	1.5038	4.9690	4.765
.205	1.5069	4.9450	4.740
.210	1.522	4.830	4.620

Hence,
$$\bar{u} = .2034,$$
$$\bar{x} = 4.984,$$
$$\bar{y} = 1.021,$$
$$\bar{x} - x_0 = 1.520.$$

In our apparatus, $\bar{y} = 2.4$, the radius of the opening of the small funnel. Then we have:

a	a	y_0	x_0	u_0	b	$\bar{u}$	$\bar{x}$	$\bar{y}$	$\bar{x} - x_0$
30°	1	2	3.464	−1.317	4.781	.2034	4.984	1.021	1.520
30°	2.35	4.7	8.14	−3.095	11.235	.4780	11.712	2.4	3.572

the second row being found from the first by multiplying throughout by $\dfrac{2.4}{1.021}\cdot$

The experimental results, in centimeters, are compared with the theoretical values in the following table:

	$2a$	$2y_0$	$\bar{x} - x_0$	$\bar{x}$
Th'l Value	4.70	9.40	3.572	11.712
Exp. I	4.7	9.45	3.65	11.78
II	4.7	9.45	3.70	11.83
III	4.7	9.45	3.67	11.80
IV	4.65	9.45	3.70	11.83
V	4.6	9.5	3.65	11.78
VI	4.7	9.5	3.70	11.83
Average	4.68	9.48	3.68	11.81
Error	0.4%	0.9%	3%	0.9%

Large experimental errors were to be expected from the lack of symmetry in the apparatus used.

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